# Kinematic constraints to the key inflationary observables

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The observables $T/S$ and $n-1$ are key to testing and understanding inflation. ($T$, $S$, and $n-1$ respectively quantify the gravity-wave and density-perturbation contributions to CMB anisotropy and the deviation of the density perturbations from the scale-invariant form.) Absent a standard model, there is no definite prediction for, or relation between, $T/S$ and $n-1$. By reformulating the equations for slow-roll inflation, we show that in the $T/S-(n-1)$ plane there are excluded regions, regions in which the density perturbations are not well approximated by a power law, and regions in which models with a ''featureless'' potential must lie.

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## INTRODUCTION

Cosmic microwave background (CMB) anisotropy measurements have begun to test inflation, the leading paradigm to extend the standard big-bang cosmology. Within a decade they should test it decisively and even probe the underlying physics [1–3]. Recent results from the BOOMERanG and MAXIMA CMB experiments [4,5] (as well as results from earlier experiments [6]) are consistent with the flat universe predicted by inflation and are beginning to address its second basic prediction: almost scale-invariant adiabatic, Gaussian density perturbations produced by quantum fluctuations during inflation [7]. The third prediction, a nearly scale-invariant spectrum of gravity waves, will be more difficult to confirm, but is a critical probe of inflation [8].

The key inflationary observables are the level of anisotropy arising from density (scalar) perturbations (quantified by the contribution to the CMB quadrupole anisotropy, $S$), the level of anisotropy arising from gravity-wave (tensor) perturbations ($T$), and the power-law index $n$ that characterizes the density perturbations (scale invariance refers to equal amplitude fluctuations in the gravitational potential on all length scales and corresponds to $n=1$). If $T$, $S$ and $n-1$ can be measured, then the scalar-field potential that drove inflation can be partially reconstructed [9]. The most promising means of measuring $T$ is its unique signature in the polarization of CMB anisotropy [10] (however, direct detection by a future space-based experiment should not be dismissed).

While there is no standard model of inflation, most models can be cast in terms of the classical evolution of a single, new scalar field $\phi$ (dubbed the inflaton) [11]. Predictions for $S$, $T$ and $n-1$ can be expressed in terms of the scalar-field potential $V(\phi)$ and its first two derivatives. While there is a model-independent relation between $T/S$ and the power-law index $n_T$ that characterizes the gravity-wave spectrum, $T/S=-5n_T$ [12,13], no such relation for $n$ and $T/S$ exists [14].

This is unfortunate because $n_T$ is very difficult to measure, while $n$ will be measured to a precision of better than 1% by the Microwave Anisotropy Probe (MAP) and Planck experiments (BOOMERanG and MAXIMA have already determined that $n\simeq 1.01^{+0.09}_{-0.07}$ [15]). Even an approximate or generic relation between $(n-1)$ and $T/S$ would be valuable, both as a test of inflation and as a guide for the expected level of gravity waves when $n$ is measured.

The formation of large-scale structure and CMB measurements already indicate that a significant part of CMB anisotropy arises from scalar perturbations, i.e. $T/S$ cannot be $\gg 1$. On the other hand, nothing precludes $T/S\ll 1$, and if $T/S$ is much less than $10^{-3}$, the prospects for measuring $T$ are poor [10]. One inflation theorist has opined that $T/S\ll 1$ for all reasonable models [16].

The goal of this work is to provide objective theoretical guidance. By casting the equations governing inflation in a form that is essentially independent of the inflaton potential (''flow equations'' for $T/S$ and $n-1$), we show that the $T/S-(n-1)$ plane is not uniformly populated by models of inflation: For $n<1$, models that are consistent with the equations governing inflation generally lie near the lines $T/S\approx 0$ and $T/S\approx -5(n-1)$, and there is an excluded region between these two lines. For $n>1$, models lie either at $T/S\approx 0$ or $T/S\approx 0.5$. Other values for $T/S$ and $n-1$ are possible, but at the expense of a spectrum of density perturbations that is poorly represented by a power law. (The CMB will be able to test how well a power law describes the density perturbations.)

## FLOW EQUATIONS

The kinematic equations that govern inflation are well known [17,18]

$$\ddot{\phi}+3H\dot{\phi}+V'(\phi)=0 \tag{1}$$

$$H^2\equiv\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi}{3m_{\rm Pl}{}^2}\left[V(\phi)+\frac{1}{2}\dot{\phi}^2\right] \tag{2}$$

where $a(t)$ is the cosmic scale factor, prime denotes $d/d\phi$, and overdot denotes $d/dt$. During inflation $\phi$ rolls slowly and the $\ddot{\phi}$ term in its equation of motion and its kinetic term in the Friedmann equation can be neglected [17,19], so that

 

$$\dot{\phi} \simeq \frac{-V'}{3H} \tag{3}$$

$$N(\phi) \equiv \int_{\phi}^{\phi_{\rm end}} H dt \simeq -\frac{8\pi}{m_{\rm Pl}^2} \int_{\phi}^{\phi_{\rm end}} \frac{d\phi}{x(\phi)} \tag{4}$$

where $x(\phi) \equiv V'(\phi)/V(\phi)$ measures the steepness of the potential and $N(\phi)$, the number of $e$-folds before the end of inflation, is the natural time variable. Inflation ends when the slow-roll conditions

$$m_{\rm Pl}|V'/V| = m_{\rm Pl}|x| < \sqrt{48\pi}, \tag{5}$$

$$m_{\rm Pl}^2|V''/V| = m_{\rm Pl}^2|x' + x^2| < 24\pi \tag{6}$$

are violated (at $\phi = \phi_{\rm end}$) [17,19].

The inflationary observables are related to the same quantities that govern the kinematics of inflation [13]

$$(n-1) = \frac{m_{\rm Pl}^2}{8\pi}[2x' - x^2] \tag{7}$$

$$T/S = \frac{5 m_{\rm Pl}^2}{8\pi} x^2 \tag{8}$$

$$T = 0.6 V/m_{\rm Pl}^4. \tag{9}$$

These expressions are given to lowest order in $x^2$ and $x'$ (see Ref. [20] for higher-order corrections). Note, $n-1$ is only equal to $n_T = -5(T/S)$ if $x' = 0$.

By combining the slow-roll equations with those governing $(n-1)$ and $T/S$, we can write equations that govern the inflationary observables (almost) without reference to a model,

$$\frac{d(T/S)}{dN} = (n-1)\frac{T}{S} + \frac{1}{5}\left(\frac{T}{S}\right)^2 \tag{10}$$

$$\frac{d(n-1)}{dN} = -\frac{1}{5}(n-1)\frac{T}{S} - \frac{1}{25}\left(\frac{T}{S}\right)^2 \pm \frac{m_{\rm Pl}^3}{16\pi^2}\sqrt{\frac{2\pi}{5}\frac{T}{S}}\, x'' \tag{11}$$

where the sign of the last term matches that of $V'$.

We call these ''flow equations'' as they describe the trajectory in the $T/S$ – $(n-1)$ plane during inflation. Because of the $x''$ term they are not completely independent of the potential. To ''close'' the flow equations we will assume that the potential is smooth enough so that we can treat $x''$ as being approximately constant. For sufficiently smooth and featureless potentials $x''$ should also be small.

Finally, one might wonder what happened to the most stringent constraint on inflation: achieving density perturbations of amplitude $10^{-5}$ or so ($S \sim 10^{-10}$). The flow equations involve the quantities $T/S$, $(n-1)$ and $dN/d\phi$ which are unaffected by a rescaling of the potential, $V \to aV$. This rescaling changes $S$: $S \to aS$. Thus, any potential can be rescaled to give proper size density perturbations without affecting the flow equations.

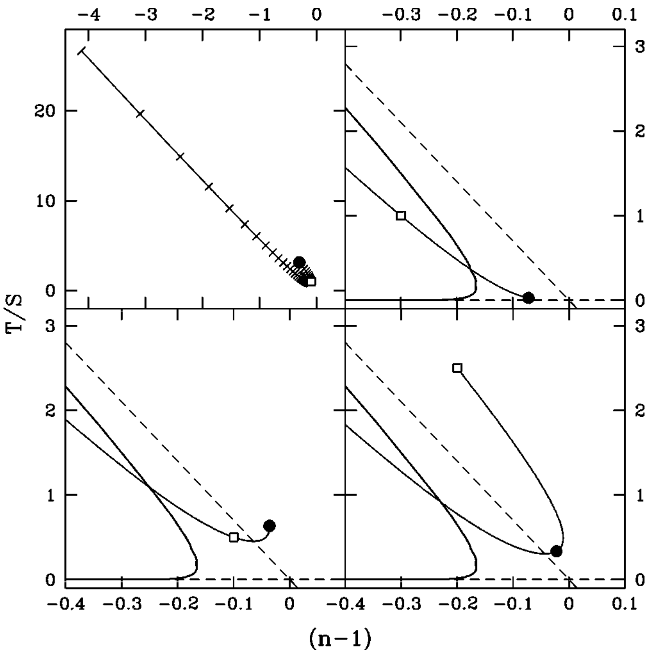


FIG. 1. Trajectories in the $T/S$ – $(n-1)$ plane. Squares indicate the initial choices for $T/S$ and $(n-1)$; circles indicate the values 50 $e$-folds before the end of inflation. A trajectory ends when $T/S$ and/or $|n-1|$ become large; most of inflation occurs when $T/S$ and $|n-1|$ are small. The upper left panel shows a complete trajectory, with ticks indicating $e$-folds before the end of inflation (from the circle, 50,49, . . . ,1). The other three panels show trajectories in more detail. Note how $T/S$ and $(n-1)$ are pulled toward the lines $T/S \approx -5(n-1)$ and $T/S \approx 0$ (these ''attractors'' are shown as broken lines and the boundary of the excluded region is a solid curve).

## THE $T/S$ – $(n-1)$ PLANE

The scales relevant for structure formation (1 Mpc to $10^4$ Mpc) crossed outside the horizon roughly 50 $e$-folds before the end of inflation (i.e., when $N=50$) [17], and so it is $T/S$ and $(n-1)$ at this time that can be measured by CMB experiments. We find them by evolving $T/S$ and $(n-1)$ until inflation ends and counting back 50 $e$-folds. To determine when inflation ends, we recast the slow-roll conditions (5),(6):

$$T/S < 30 \tag{12}$$

$$\left|(n-1) + \frac{3}{5}\frac{T}{S}\right| < 6; \tag{13}$$

the violation of either indicates the end of inflation.

To be very specific about our procedure, we choose a starting point for our calculation in the range, $0 < T/S < 10$ and $-0.5 < (n-1) < 0.5$. Our results do not depend upon the range of these initial values. We then integrate with fixed $x''$ until one of the slow-roll conditions is violated, signaling the end of inflation, and count back 50 $e$-folds to find $(T/S)_{50}$ and $(n-1)_{50}$. Some trajectories are shown in Fig. 1.

Figures 2 and 3 summarize the $(T/S)_{50}$ – $(n-1)_{50}$ phase space generated from the range of initial conditions considered. It is not uniformly populated. For $x'' < \mathcal{O}(1)$, solutions cluster around two ''attractors,'' $(T/S)_{50} \approx 0$ and $(T/S)_{50} \approx -5(n-1)_{50}$, and for $(n-1)_{50} < 0$, there is an excluded re-

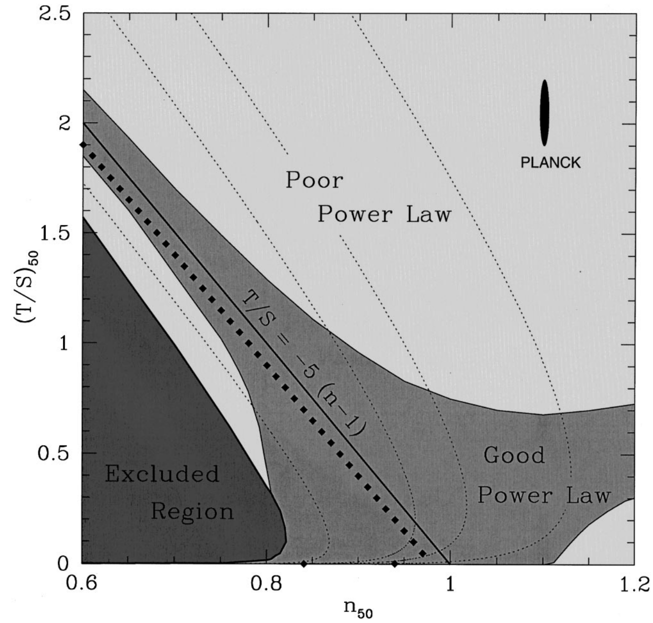


FIG. 2. Summary of our model search using the flow equations. The dotted curves correspond to $x''=0,1,2,5$ (from left to right). We found no models in the excluded region. The good power law region represents the models for which the density perturbation spectrum closely follows a power law. (Specifically, $|dn/d\ln k|<10^{-2}$.) Diamonds indicate various known inflationary models: chaotic, $V(\phi)=\lambda\phi^n$ for $n=2,3,\ldots$ (diamonds on the diagonal); new inflation ($n=0.94$) and natural inflation (with $n=0.84$). The ellipse is the $2\sigma$ error ellipse ''forecasted'' for the Planck satellite [21].

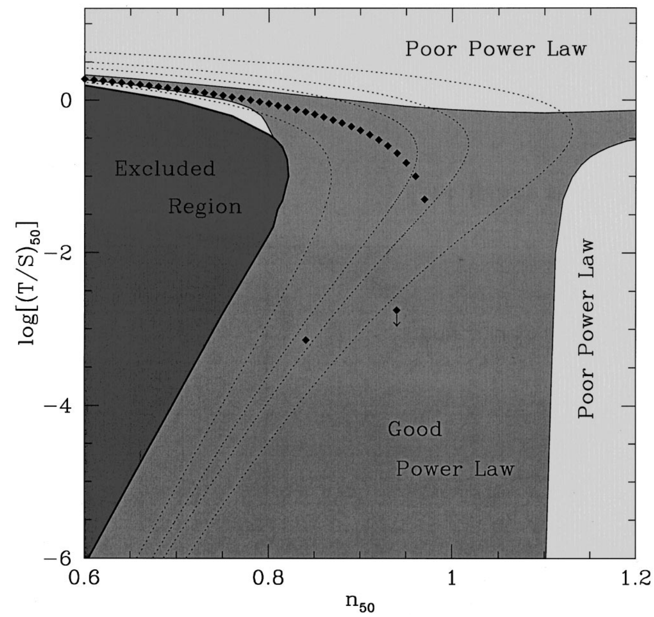


FIG. 3. Same as Fig. 2, with a logarithmic scale for $T/S$.

gion between these two lines, which cannot be reached for any value of $x''$, i.e. there are *no* models in this excluded region. Noting that $dn/d\ln k=-dn/dN$ and using Eq. (11), we can calculate the running of the scalar index. We define a good-power-law region, for which $|dn/d\ln k|<10^{-2}$; models outside of this region will have a poor power law, a prediction that can be tested by CMB measurements. The results shown in Figs. 2 and 3 do not depend strongly on the value of $N$ chosen for our calculation. For values of $N$ taken between 40 and 60, the borders of the regions shift slightly but not significantly.

Taking $x''=0$ it is simple to show why there are no models in the excluded region of the $T/S-(n-1)$ plane. In this limit, the flow equations are: $s\equiv(n-1)+\frac{1}{5}(T/S)=$ const and $r\propto\exp(sN)$, where $r=T/S$. Unless $r$ and/or $s$ are small, corresponding to the attractor solutions, $r$ grows very rapidly and inflation does not last 50 $e$-folds.

Models outside of our good-power-law region are possible, but they come at the expense of a density-perturbation spectrum that is not well represented by a power law. Moreover, they are characterized by large $x''$, which typically signals that the potential has a feature, e.g. a ''bump'' or a ''kink.'' We note that it is possible to have large $x''$ but still have a good power law if $T/S\ll 1$. This explains the results of a recent paper [22] in which models with $n$ as large as 2 were constructed. In particular, for the model with $n=2$, $x''\simeq 2000$, $T/S\simeq 3\times 10^{-3}$ and $dn/d\ln k\simeq 0.3$.

So far, we have only considered one-field inflation. There are also models with more than one field, either implicitly or explicitly [23]. However, in essentially all models discussed in the literature only one field plays an ''active'' role during inflation [11]. The other field(s) are used to halt inflation and make a graceful exit to a radiation-dominated cosmology (e.g., by classical evolution in hybrid inflation or a phase

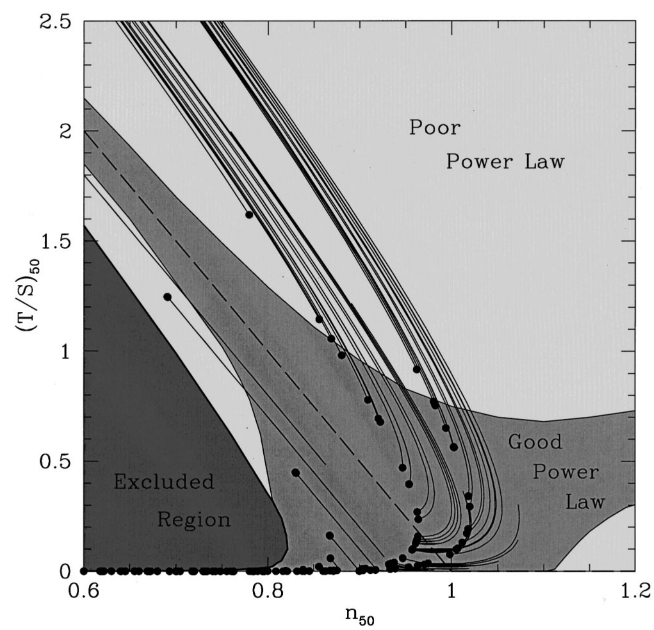


FIG. 4. Summary of two-field models. The filled circles represent the values of $(T/S)_{50}$ and $n_{50}$ for the corresponding one-field models, and the attached curves are the values obtained for inflation ending early due to an auxiliary field. The dashed lines represent fixed points in the $(T/S)-(n-1)$ plane that result from models that do not end without an auxiliary field. In general, two-field models populate the same region as one-field models and extend the $T/S\approx 0$ part of the good-power-law region to $n>1$.

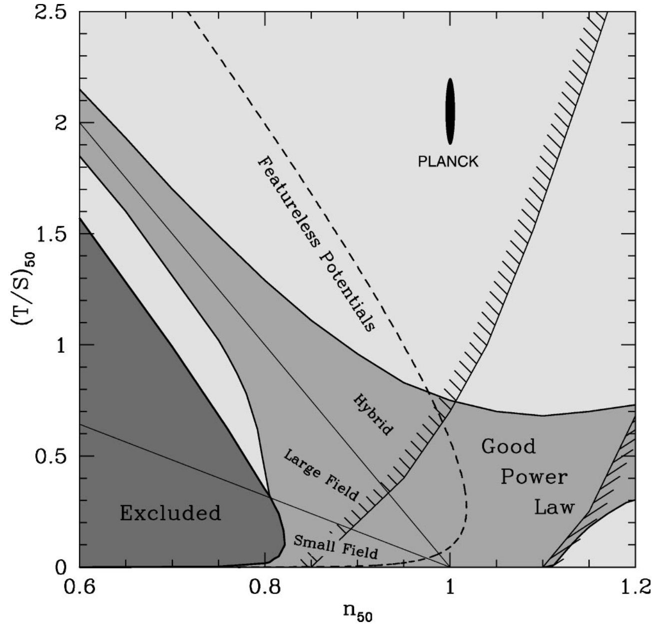


FIG. 5. Features of the $T/S$ – $n$ plane. There is an excluded region in the lower left corner of the plane in which there are no models. Inflationary models that result in a scalar perturbation spectrum that is well represented by a power law are found in the good-power-law region. We have defined the borders of this region as $|dn/d\ln k|<10^{-2}$. The area to the left of the dashed line contains models that result from featureless potentials, i.e. potentials with small $x''$ (specifically $x''<2$). We have also shown the division of the $T/S$ – $n$ plane into ''small field,'' ''large field,'' and ''hybrid'' theoretical models advocated in Ref. [24] along with the 99% confidence level constraints from recent CMB data [24]. The allowed region is within the hatched lines. The Planck ellipse is the same as in Fig. 2.

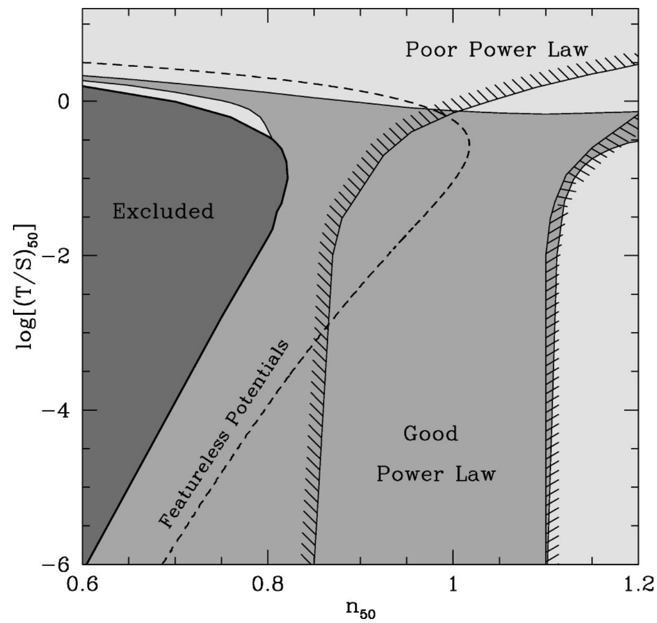


FIG. 6. Same as Fig. 5, with a logarithmic scale for $T/S$.

transition in extended inflation). The most well known of these is power-law inflation, $V(\phi)\propto\exp(-\beta\phi/m_{\rm Pl})$, a model in which inflation would not end in the absence of the action of another field. Our flow equations can also be applied to such multi-field models.

Models that require another field to end inflation show up when the right-hand sides of Eqs. (10),(11) vanish prior to violating the slow-roll conditions. In this case, there are fixed points in the $T/S$ – $(n-1)$ plane, which are the most likely values for $T/S$ and $(n-1)$ 50 $e$-folds prior to when the second field ends inflation. These points, shown in Fig. 4, populate the regions $T/S\approx0$ for $n>1$ and the line $T/S=-5(n-1)$ for $n<1$.

It is also possible that a self-ending model has an auxiliary field that ends inflation ''early.'' We treat this possibility by populating the $(T/S)_{50}$ – $(n-1)_{50}$ plane with the values of $T/S$ and $(n-1)$ at $N>50$ for all one-field models. We find that the two-field models behave similarly to the one-field models. The only significant difference is that two-field models extend the $(T/S)_{50}\approx0$ part of the good-power-law region to $(n-1)_{50}>0$ (see Fig. 4).

Finally, what about our taking $x''\simeq$const? It can affect the relationship between the initial and final values of $n-1$ and $T/S$ if $x''$ is large, since $x''$ need not be constant (as is the case in some known models). Since we have covered a wide range of initial values we would expect that this fact would only slightly modify the $(n-1)_{50}$ – $(T/S)_{50}$ phase space; indeed, we have also formulated the flow equations assuming $V'''/V=$const and obtain similar results.

## DISCUSSION

Prior to this work there was one guiding relation for the inflationary observables: $T/S=-5n_T$. It has the virtue of exactitude and can test the consistency of the scalar-field inflationary framework, but it involves the power-law index of the gravity-wave perturbations, the most difficult observable to measure. By reformulating the equations governing inflation, we have found generic relations between $T/S$ and $(n-1)$ which are summarized in Figs. 5 and 6 and below:

(i) For $n<1$ the good-power-law region in the $T/S$ – $(n-1)$ plane has $T/S\approx-5(n-1)$ or 0.

(ii) For $n>1.1$ the good-power-law region in the $T/S$ – $(n-1)$ plane has $0.25<T/S<0.7$ (one-field models) or 0 (two-field models).

(iii) For $0.85<n<1$ and $|x''|<2$ (featureless potentials), $T/S>10^{-3}$.

(iv) For $n<0.82$ there is a large excluded region in the $T/S$ – $(n-1)$ plane in which absolutely *no* models are found.

(v) There is a correlation between the values of $dn/d\ln k$, $T/S$, and $n$.

Our results provide some guidance to CMB experimenters looking for tensor perturbations [cf. (ii) and (iii)] and additional consistency tests for inflation [cf. (iv) and (v)].

## ACKNOWLEDGMENTS

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